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MULTIDIMENSIONAL GRAVITY WITH EINSTEIN INTERNAL SPACES

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Abstract

A multidimensional gravitational model on the manifold $M = M_0 \times \prod_{i=1}^{n} M_i$, where M_i are Einstein spaces ($i \geq 1$), is studied. For $N_0 = \dim M_0 > 2$ the σ model representation is considered and it is shown that the corresponding Euclidean Toda-like system does not satisfy the Adler-van-Moerbeke criterion. For $M_0 = \mathbf{R}^{N_0}$, $N_0 = 3, 4, 6$ (and the total dimension $D = \dim M = 11, 10, 11$, respectively) nonsingular spherically symmetric solutions to vacuum Einstein equations are obtained and their generalizations to arbitrary signatures are considered. It is proved that for a non-Euclidean signature the Riemann tensor squared of the solutions diverges on certain hypersurfaces in $\mathbf{R}^{N_0}$.

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1 Introduction

Our paper is devoted to studying a model of multidimensional gravity considered previously in Refs. [1–3] (see also [24–27]). This model contains "our space" M_0 of dimension N_0 and a set of internal Einstein spaces $M_1, \ldots, M_n$. All scale factors of M_i are supposed to be functions on M_0. For physical applications $N_0 \leq 4$ (e.g $N_0 = 1, 2$ corresponds to cosmology and axial symmetry, respectively).

On the classical level the model is equivalent to some tensor-multiscalar theory and may be also treated as a generalization of the standard Brans-Dicke theory with the parameter $\omega = 1/N' - 1$, where N' is the total internal space dimension [3].

It should be noted that scalar-tensor theories are rather popular now (see, for example [4]-[8]).

For N_0=1 we get a multidimensional cosmological model considered by many authors [10]-[42]. This model was reduced to a pseudo-Euclidean Toda-like system [29, 32, 36, 40], which is a rather nontrivial object, since there are no explicit integration methods or integrability conditions when the number of spaces with nonzero curvature is greater than one. Recently, in Ref. [41] three integrable non-trivial families of solutions were obtained for a cosmological model with two nonzero curvatures ($n = 2$) and $(N_1 = \dim M_1, N_2 = \dim M_2) = (3, 6), (5, 5), (2, 8)$ by solving the Abel equation. They include nonsingular spherically symmetric solutions on manifolds $\mathbf{R}^7 \times M_2$ [41] and $\mathbf{R}^6 \times M_2$ [42] for $\dim M = 3$ and 5, respectively. As it is hard to solve the Abel equation from [41] for arbitrary (N_1, N_2), we may first try to obtain nonsingular spherically symmetric solutions on $\mathbf{R}^{N_0} \times M_2$ and then try to extend them to the general solution for a cosmology with two curvatures on the manifold $\mathbf{R}_+ \times \mathbf{S}^{N_0-1} \times M_2$.

The paper is organized as follows. In Sec. 2 we describe the model and obtain the equations of motion. In Sec. 3 the non-exceptional case $N_0 \neq 2$ is considered. We obtain a generalized σ model and in the case $N_0 > 2$ (such that the "midisuperspace" metric is Euclidean) show that the interaction potential does not satisfy the Adler-van-Moerbeke criterion [44]. We diagonalize the "midisuperspace" metric and obtaine a "diagonalized" σ model representation in a more explicit manner than in [1, 3]. In Sec. 4 three families of nonsingular spherically symmetric solutions with the topology $\mathbf{R}^{N_0} \times M_1 \times \ldots \times M_n$ are obtained for $N_0 = 3, 4, 6$ and the total dimension $D = 11, 10, 11$, respectively. (We thus obtain as well one exact solution for ten-dimensional superstring gravity [50] and two solutions for eleven-dimensional supergravity [51] and M-theory [52].) These solutions are generalized to arbitrary signatures of the N_0-dimensional section of the metric. The Riemann tensor squared for the solutions is calculated and it is proved that for non-Euclidean signatures it is divergent on a certain (generalized) hypersphere in $\mathbf{R}^{N_0}$. An example of a de-Sitter membrane solution is suggested. In Sec. 5 we consider the exceptional case $N_0 = 2$. We show that in this case the midisuperspace metric is not uniquely determined and depends on the choice of the conformal frame. (As pointed out in Ref. [3], there is no conformal Einstein-Pauli frame in this case). Two examples corresponding to different conformal frames are presented.

2 The model

Let us consider the manifold

$$M = M_0 \times M_1 \times \ldots \times M_n, \tag{2.1}$$

with the metric

$$g = \mathrm{e}^{2\gamma(x)} g^0 + \sum_{i=1}^{n} \mathrm{e}^{2\phi^i(x)} g^i, \tag{2.2}$$

where

$$g^0 = g^0_{\mu\nu}(x) dx^\mu \otimes dx^\nu \tag{2.3}$$

is a metric on the manifold M_0 and g^i is a metric on M_i satisfying the equation

$$R_{m_i n_i}[g^i] = \lambda_i g^i_{m_i n_i}, \tag{2.4}$$

$m_i, n_i = 1, \ldots, N_i$; $\lambda_i =$ const, $i = 1, \ldots, n$. Thus (M_i, g^i) are Einstein spaces. The functions $\gamma, \phi^i : M_0 \to \mathbf{R}$ are smooth.

Remark 1. It is more correct to write (2.2) as

$$g = \exp[2\gamma(x)]\hat{g}^0 + \sum_{i=1}^{n} \exp[2\phi^i(x)]\hat{g}^i$$

where we denote by $\hat{g}^\alpha = p^*_\alpha g^\alpha$ the pullback of the metric g^α to the manifold M by the canonical projection: $p_\alpha : M \to M_\alpha$, $\alpha = 0, \ldots, n$. In what follows all "hats" over metrics will be omitted.

Here we are interested in exact solutions to the Einstein equations with a cosmological constant

$$R_{MN}[g] - \tfrac{1}{2} g_{MN} R[g] = -\Lambda g_{MN} \tag{2.5}$$

for the metric (2.2) defined on the manifold (2.1). The set of equations (2.5) is equivalent to

$$R_{MN}[g] = \frac{2\Lambda}{D-2} g_{MN}, \tag{2.6}$$

where $D = \sum_{k=0}^{n} N_k = \dim M$ is the dimension of the manifold (2.1), $N_k = \dim M_k$, $k = 0, \ldots, n$. Eqs. (2.5) are the field equations corresponding to the action

$$\begin{aligned} S &= S[g] \\ &= \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|g|} \{R[g] - 2\Lambda\} + S_{\mathrm{GH}} \end{aligned} \tag{2.7}$$

where we denote $|g| = |\det(g_{MN})|$: $S_{\rm GH}$ is the standard Gibbons-Hawking boundary term [43]. This term is essential for a quantum treatment of the problem.

The nonzero Ricci tensor components for the metric (2.2) are (see the Appendix)

$$\begin{aligned} R_{\mu\nu}[g] = R_{\mu\nu}[g^0] + g^0_{\mu\nu}\Bigl[-\Delta_0\gamma + (2-N_0)(\partial\gamma)^2 \\ - \partial\gamma \sum_{j=1}^{n} N_j \partial\phi^j\Bigr] + (2-N_0)(\gamma_{;\mu\nu} - \gamma_{,\mu}\gamma_{,\nu}) \\ - \sum_{i=1}^{n} N_i(\phi^i_{;\mu\nu} - \phi^i_{,\mu}\gamma_{,\nu} - \phi^i_{,\nu}\gamma_{,\mu} + \phi^i_{,\mu}\phi^i_{,\nu}), \end{aligned} \quad (2.8)$$

$$\begin{aligned} R_{m_i n_i}[g] = R_{m_i n_i}[g^i] - \mathrm{e}^{2\phi^i - 2\gamma} g^i_{m_i n_i}\Bigl\{\Delta_0\phi^i \\ + (\partial\phi^i)[(N_0-2)\partial\gamma + \sum_{j=1}^{n} N_j\partial\phi^j]\Bigr\}, \end{aligned} \quad (2.9)$$

Here $\partial\beta\,\partial\gamma \equiv g^{0\ \mu\nu}\beta_{,\mu}\gamma_{,\nu}$ and Δ_0 is the Laplace-Beltrami operator corresponding to g^0. The scalar curvature for (2.2) is

$$\begin{aligned} R[g] = \sum_{i=1}^{n} \mathrm{e}^{-2\phi^i} R[g^i] + \mathrm{e}^{-2\gamma}\Bigl\{R[g^0] - \sum_{i=1}^{n} N_i(\partial\phi^i)^2 \\ - (N_0-2)(\partial\gamma)^2 - (\partial f)^2 - 2\Delta_0(f+\gamma)\Bigr\} \end{aligned} \quad (2.10)$$

where

$$f = f(\gamma,\phi) = (N_0-2)\gamma + \sum_{j=1}^{n} N_j\phi^j. \quad (2.11)$$

Using (2.8) and (2.9), it is not difficult to verify that the field equations (2.5) (or, equivalently, (2.6)) may be obtained as the equations of motion corresponding to the action

$$\begin{aligned} & S_\sigma[g^0,\gamma,\phi] \\ & = \frac{1}{2\kappa_0^2}\int_{M_0} d^{N_0}x\sqrt{|g^0|}\,\mathrm{e}^{f(\gamma,\phi)}\Bigl\{R[g^0] \\ & - \sum_{i=1}^{n} N_i(\partial\phi^i)^2 - (N_0-2)(\partial\gamma)^2 + (\partial f)\partial(f+2\gamma) \\ & + \sum_{i=1}^{n}\lambda_i N_i\,\mathrm{e}^{-2\phi^i+2\gamma} - 2\Lambda\,\mathrm{e}^{2\gamma}\Bigr\}. \end{aligned} \quad (2.12)$$

where $|g^0| = |\det(g^0_{\mu\nu})|$ and similar notations are applied to the metrics g^i, $i = 1,\ldots,n$. For finite internal space volumes (e.g. compact M_i)

$$V_i = \int_{M_i} d^{N_i}y\sqrt{|g^i|} < +\infty, \quad (2.13)$$

the action (2.12) coincides with the action (2.7), i.e.

$$S_\sigma[g^0,\gamma,\phi] = S[g], \quad (2.14)$$

where g is defined by the relation (2.2) and

$$\kappa^2 = \kappa_0^2\prod_{i=1}^{n} V_i. \quad (2.15)$$

This may be readily verified using the following relation for the scalar curvature (2.10):

$$\begin{aligned} R[g] = \sum_{i=1}^{n} \mathrm{e}^{-2\phi^i} R[g^i] + \mathrm{e}^{-2\gamma}\Bigl\{R[g^0] - \sum_{i=1}^{n} N_i(\partial\phi^i)^2 \\ - (N_0-2)(\partial\gamma)^2 + (\partial f)\partial(f+2\gamma) + R_B\Bigr\}, \end{aligned} \quad (2.16)$$

where

$$R_B = (1/\sqrt{|g^0|})\,\mathrm{e}^{-f}\partial_\mu[-2\,\mathrm{e}^f\sqrt{|g^0|}g^{0\ \mu\nu}\partial_\nu(f+\gamma)] \quad (2.17)$$

gives rise to the Gibbons-Hawking boundary term

$$S_{\rm GH} = \frac{1}{2\kappa^2}\int_M d^Dx\sqrt{|g|}\{-\mathrm{e}^{-2\gamma}R_B\}. \quad (2.18)$$

3 The non-exceptional case $N_0 \neq 2$

In order to simplify the action (2.12), we use, as in [1] for $N_0 \neq 2$, the gauge

$$\gamma = \gamma_0(\phi) = \frac{1}{2-N_0}\sum_{i=1}^{n} N_i\phi^i. \quad (3.1)$$

It means that $f = f(\gamma_0,\phi) = 0$, or the conformal Einstein-Pauli frame is used. Evidently this frame does not exist for $N_0 = 2$. For the cosmological case $N_0 = 1$, $g^0 = -dt\otimes dt$, and (3.1) corresponds to the harmonic-time gauge [29]. ¿From (3.1) we get

$$\begin{aligned} & S_0[g^0,\phi] = S_\sigma[g^0,\gamma_0,\phi] = \\ & \frac{1}{2\kappa_0^2}\int_{M_0} d^{N_0}x\sqrt{|g^0|}\Bigl\{R[g^0] \\ & - G_{ij}g^{0\ \mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j - 2V(\phi)\}, \end{aligned} \quad (3.2)$$

where

$$G_{ij} = N_i\delta_{ij} + \frac{N_iN_j}{N_0-2} \quad (3.3)$$

are the components of the "midisuperspace" (or target space) metric on $\mathbf{R}^n$

$$G = G_{ij}d\phi^i\otimes d\phi^j \quad (3.4)$$

and

$$V = V(\phi) = \Lambda\,\mathrm{e}^{2\gamma_0(\phi)} - \tfrac{1}{2}\sum_{i=1}^{n}\lambda_i N_i\,\mathrm{e}^{-2\phi^i+2\gamma_0(\phi)} \quad (3.5)$$

is the potential. (Here we corrected a misprint in Eq. (11) from [1].) Thus, we are led to the action of a self-gravitating σ model with a flat target space $(\mathbf{R}^n, G)$ (3.4) and a self-interaction described by the potential (3.5).

For $N_0 = 1$, $g^0 = -dt \otimes dt$ the action (3.2) coincides with the well-known cosmological one [29]. In this case the minisuperspace metric (3.3) is pseudo-Euclidean [27, 29].

Remark 2. We note that in the infinite-dimensional case $n = \infty$ the potential (3.5) is well-defined if the following restrictions are imposed:

$$\sum_{i=1}^{n} |\lambda_i| N_i < +\infty, \qquad \sum_{i=1}^{n} N_i |\phi^i| < +\infty. \tag{3.6}$$

In the case $N_0 = 1$, $\phi = (\phi^i)$ belongs to a Banach space with l_1-norm [32].

3.1. The case $N_0 > 2$

For $N_0 > 2$ the midisuperspace metric (3.3) is Euclidean. The potential (3.5) may be rewritten as

$$V(\phi) = \sum_{\alpha=0}^{n} A_\alpha \exp[u_i^\alpha \phi^i], \tag{3.7}$$

including the cosmological constant and the curvature terms, where $A_0 = \Lambda$, $A_j = -\frac{1}{2}\lambda_j N_j$ and

$$u_i^0 = \frac{2N_i}{2 - N_0}, \qquad u_i^j = 2\Big(-\delta_i^j + \frac{N_i}{2 - N_0}\Big), \tag{3.8}$$

$i, j = 1, \ldots, n$. Thus the potential (3.5) has a Toda-like form.

Let

$$\langle u, v \rangle_* \equiv G^{ij} u_i v_j \tag{3.9}$$

be a quadratic form on $\mathbf{R}^n$. Here

$$G^{ij} = \frac{\delta_{ij}}{N_i} + \frac{1}{2 - D} \tag{3.10}$$

are components of the matrix inverse to the matrix (G_{ij}) in (3.3). For the vectors (3.8) $u^\alpha = (u_i^\alpha) \in \mathbf{R}^n$, $\alpha = 0, \ldots, n$, we get the following relations:

$$\langle u^0, u^0 \rangle_* = \frac{4(D - N_0)}{(N_0 - 2)(D - 2)}, \tag{3.11}$$

$$\langle u^0, u^j \rangle_* = \frac{4}{(N_0 - 2)}, \tag{3.12}$$

$$\langle u^i, u^j \rangle_* = 4\bigg(\frac{\delta_{ij}}{N_i} + \frac{1}{N_0 - 2}\bigg), \tag{3.13}$$

$i, j = 1, \ldots, n$.

3.2. The Adler-van-Moerbeke criterion

For a fixed metric g^0 the action (3.2) coincides with the action of a Euclidean Toda-like system, i.e. a dynamical (physical) system with the potential in the form of a sum of exponents depending on linear combinations of cordinates (fields). For Toda-like systems in the dimension $N_0 = 1$ [47]-[49] (with the appropriate number of exponents) we know that the integrable cases (open and closed Toda lattices) occur when the vectors u^α in the exponents correspond to roots of an appropriate finite-dimensional semisimple Lie algebra or an infinite-dimensional affine Lie algebra.

This situation may be described by the so-called Adler-van-Moerbeke criterion [44]. Here we formally extend this criterion to the case $N_0 > 1$ and apply it to our model with a fixed metric g^0.

When all $A_\alpha \neq 0$ in (3.5) and the vectors u^α satisfy the Adler-van-Moerbeke criterion [44],

$$K_{\alpha\beta} \equiv \frac{2\langle u^\alpha, u^\beta \rangle_*}{\langle u^\beta, u^\beta \rangle_*} = \hat{C}_{\alpha\beta}, \tag{3.14}$$

$\alpha = 0, \ldots, n$, where $\hat{C} = (\hat{C}_{\alpha\beta})$ is the Cartan matrix corresponding to some affine Lie algebra $\hat{\mathcal{G}}$ [45], then the considered Toda-like system (3.2) with fixed g^0 is equivalent to an N_0-dimensional closed Toda lattice on (M_0, g^0) corresponding to $\hat{\mathcal{G}}$.

When $\Lambda = 0$, $\lambda_i \neq 0$, $i = 1, \ldots, n$, $n \geq 2$ and

$$K_{ij} = C_{ij}, \tag{3.15}$$

$i, j = 1, \ldots, n$, where $C = (C_{ij})$ is the Cartan matrix corresponding to some semisimple Lie algebra $\mathcal{G}$ of rank n, then the Toda-like system (3.2) with fixed g^0 is equivalent to an N_0-dimensional open Toda lattice on (M_0, g^0) corresponding to $\mathcal{G}$.

Now, we show that the relations (3.14) and (3.15) are not satisfied for $N_i \in \mathbf{N}$ ($N_i > 1$, since $\lambda_i \neq 0$), $i = 1, \ldots, n$, $n \geq 2$. Indeed, from (3.13) we get

$$K_{ij} = \frac{2\big[\delta_{ij}/N_j + 1/(N_0 - 2)\big]}{1/N_j + 1/(N_0 - 2)} > 0, \tag{3.16}$$

It follows from (3.16) that the relation (3.15) is never satisfied for $N_i \in \mathbf{N}$, since

$$C_{ij} = -n_{ij}, \qquad n_{ij} \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}, \tag{3.17}$$

for $i \neq j$ ($n_{ij} = 0, 1, 2, 3$). For the same reason ($\hat{C}_{\alpha\beta} \leq 0, \alpha \neq \beta$) the relation (3.14) is never satisfied for positive integers N_j and $n \geq 1$ (see (3.12)). Thus, the model under consideration (3.2) (with fixed g^0) is not equivalent to an N_0-dimensional (closed or open) Toda lattice (when the number of nonzero terms in the potential (3.5) is greater than one) and seems to be a rather nontrivial object of non-linear analysis.

Remark 3. If we consider (at least formally) the model (3.2) with $\Lambda = 0$ and complex dimensions N_j, $j = 1, \ldots, n$, obeying the restriction

$$\det(G_{ij}) = N_1 \ldots N_n \frac{2 - D}{2 - N_0} \neq 0, \tag{3.18}$$

then we find the following solution of (3.15): $n = 2$,

$$\{N_1, N_2\} = \bigg\{\frac{1}{3}(2 - N_0), \frac{k}{k + 2}(2 - N_0)\bigg\} \tag{3.19}$$

$k = 1, 2, 3$, corresponding to the Lie algebras $a_2 = sl(3)$, $b_2 = so(5)$ and g_2, respectively. (The cosmological case $N_0 = 1$ was considered earlier in Ref. [32]. For $N_0 = 1$, $k = 1$ see also Ref. [23].)

3.3. Diagonalization

The case $N_0 > 2$**.** Let us diagonalize the midisupermetric. This may be useful for quantization of the σ model under study. For $N_0 > 2$ the midisupermetric may be diagonalized by the linear transformation

$$\varphi^a = S^a_i \phi^i, \tag{3.20}$$

where

$$S^a_i \delta_{ab} S^b_j = G_{ij}, \tag{3.21}$$

$a,b = 1,\dots,n;\ i,j = 1,\dots,n$. Then Eq. (3.4) reads:

$$G = \delta_{ab} d\varphi^a \otimes d\varphi^b. \tag{3.22}$$

An example of diagonalization (3.20), (3.21) is

$$\varphi^1 = q^{-1} \sum_{i=1}^{n} N_i \phi^i, \tag{3.23}$$

$$\varphi^{\hat{b}} = \Bigl[N_{\hat{b}-1}/(\Sigma_{\hat{b}-1}\Sigma_{\hat{b}}) \Bigr]^{1/2} \sum_{j=\hat{b}}^{n} N_j (\phi^j - \phi^{\hat{b}-1}), \tag{3.24}$$

$\hat{b} = 2,\dots,n$, where

$$q = q(N_0, D) \equiv \Bigl[\frac{(D-N_0)|N_0-2|}{(D-2)} \Bigr]^{1/2}, \ \Sigma_a \equiv \sum_{j=a}^{n} N_j. \tag{3.25}$$

Consider a more general class of the diagonalization (3.20) satisfying (3.23) or, equivalently,

$$S^1_i = q^{-1} N_i, \tag{3.26}$$

Let us introduce

$$S^a = (S^a_i) \in \mathbf{R}^n, \tag{3.27}$$

$a = 1,\dots,n$. The relation (3.21) is equivalent to

$$S^a_i G^{ij} S^b_j = \langle S^a, S^b \rangle_* = \delta^{ab}. \tag{3.28}$$

For $a,b = 1$ the relation (3.28) is satisfied identically due to (3.25) and (3.26) (see also (3.8), (3.11)). For $\hat{b} > 1$

$$0 = \langle S^1, S^{\hat{b}} \rangle_* = q^{-1} N_i G^{ij} S^{\hat{b}}_j = q^{-1} \frac{2-N_0}{2-D} \sum_{j=1}^{n} S^{\hat{b}}_j, \tag{3.29}$$

or, equivalently,

$$0 = \sum_{j=1}^{n} S^{\hat{b}}_j. \tag{3.30}$$

Here we use the relation

$$G^{ij} N_j = \frac{2-N_0}{2-D}. \tag{3.31}$$

For $\hat{a}, \hat{b} > 1$ we get from (3.30)

$$\delta^{\hat{a}\hat{b}} = \langle S^{\hat{a}}, S^{\hat{b}} \rangle_* = \Bigl(\frac{\delta_{ij}}{N_i} + \frac{1}{2-D} \Bigr) S^{\hat{a}}_i S^{\hat{b}}_j = \frac{\delta_{ij}}{N_i} S^{\hat{a}}_i S^{\hat{b}}_j \tag{3.32}$$

or, equivalently,

$$\sum_{i=1}^{n} \frac{1}{N_i} S^{\hat{a}}_i S^{\hat{b}}_i = \delta^{\hat{a}\hat{b}}. \tag{3.33}$$

Thus, when the condition (3.26) is imposed, the relation (3.21) is equivalent to the set of relations (3.30), (3.33). It is not difficult to verify that these relations are satisfied for $(S^{\hat{a}}_i)$ from (3.24). For the inverse matrix we get from (3.28)

$$\hat{S}^i_a = G^{ij} S^b_j \delta_{ba} = G^{ij} S^a_j \tag{3.34}$$

and, hence, (see (3.26) and (3.31))

$$\hat{S}^i_1 = G^{ij} S^1_j = q^{-1} \frac{2-N_0}{2-D} = \frac{q}{D-N_0}. \tag{3.35}$$

¿From the relation

$$\hat{S}^i_a G_{ij} \hat{S}^j_b = \delta_{ab} \tag{3.36}$$

(following from (3.28)) and Eqs. (3.10), (3.35), (3.36) we get

$$\sum_{j=1}^{n} N_j \hat{S}^j_{\hat{b}} = 0, \qquad \sum_{i=1}^{n} N_i \hat{S}^i_{\hat{a}} \hat{S}^i_{\hat{b}} = \delta_{\hat{a}\hat{b}}, \tag{3.37}$$

$\hat{a}, \hat{b} > 1$. Here we have used the relation

$$\sum_{i=1}^{n} G_{ij} = N_j \frac{D-2}{N_0-2}. \tag{3.38}$$

In the new variables (3.20) satisfying (3.26) the action (3.2) reads:

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{N_0}x \sqrt{|g^0|} \{ R[g^0] - \delta_{ab} g^{0\ \mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b - 2V \}. \tag{3.39}$$

where

$$V = \sum_{\alpha=0}^{n} A_\alpha \exp[\hat{u}^\alpha_a \varphi^a]. \tag{3.40}$$

Here the following notation is used:

$$\hat{u}_a = S^i_a u_i. \tag{3.41}$$

It follows from (3.35) that

$$\hat{u}_1 = \hat{S}^i_1 u_i = \frac{q}{D-N_0} \sum_{i=1}^{n} u_i. \tag{3.42}$$

For the vectors (3.8), corresponding to the Λ-term and the curvature components, respectively, we have

$$\hat{u}^0_1 = \frac{2q}{2-N_0}, \qquad \hat{u}^j_1 = -2q^{-1}, \tag{3.43}$$

$j = 1, \ldots, n$. We denote $\vec{u}_* = (\hat{u}_2, \ldots, \hat{u}_n)$. Then $\vec{u}^0_* = 0$ (see (3.37)) and

$$\vec{u}^i_* \vec{u}^j_* = \langle u^i, u^j \rangle_* + 4q^{-2} = 4\Big(\frac{\delta_{ij}}{N_i} + \frac{1}{N_0 - D}\Big), \tag{3.44}$$

$i, j = 1, \ldots, n$ (see (3.13), (3.43)). Thus the potential (3.40) (see (3.5)) may be written as

$$V = \Lambda \exp\Big[\frac{2q\varphi^1}{2 - N_0}\Big] + \exp(-2q^{-1}\varphi^1) V_*(\vec{\varphi}_*) \tag{3.45}$$

where

$$V_*(\vec{\varphi}_*) = \sum_{i=1}^{n} (-\frac{1}{2}\lambda_i N_i) \exp(\vec{u}^i_* \vec{\varphi}_*), \tag{3.46}$$

$\vec{\varphi}_* = (\varphi_2, \ldots, \varphi_n)$ and the vectors $\vec{u}^i_* \in \mathbf{R}^{n-1}$ satisfy the relations (3.44).

The cosmological case $N_0 = 1$**.** In the cosmological case $M_0 = \mathbf{R}$, $g^0 = -\mathcal{N}^2(t) dt \otimes dt$, ($\mathcal{N}(t) > 0$ is the lapse function) for the metric (2.2)

$$g = -\,\mathrm{e}^{2\gamma_0(t)} \mathcal{N}^2(t) dt \otimes dt + \sum_{i=1}^{n} \mathrm{e}^{2\phi^i(x)} g^i \tag{3.47}$$

the action (3.2) reads [29]:

$$S = S[\mathcal{N}, \phi] = \frac{1}{\kappa_0^2} \int dt \mathcal{N} \{ \tfrac{1}{2} \mathcal{N}^{-2} \bar{G}_{ij} \dot{\phi}^i \dot{\phi}^j - V(\phi) \}, \tag{3.48}$$

where

$$\bar{G}_{ij} = N_i \delta_{ij} - N_i N_j \tag{3.49}$$

are components of a pseudo-Euclidean minisuperspace metric on $\mathbf{R}^n$ and the potential V is defined in (3.5).

Let us consider the diagonalization

$$\varphi^a = S^a_i \phi^i, \qquad S^a_i \eta_{ab} S^b_j = G_{ij}, \tag{3.50}$$

($(\eta^{ab}) = \mathrm{diag}(-1, 1, \ldots, 1)$, $a, b = 0, \ldots, n-1$; $i, j = 1, \ldots, n$) satisfying Eq. (3.26) with q from (3.25) ($N_0 = 1$). Just as before, it may be shown that in the new variables φ^a the action (3.48) has the form

$$S = S[\mathcal{N}, \phi] = \frac{1}{\kappa_0^2} \int dt \mathcal{N} \{ \tfrac{1}{2} \mathcal{N}^{-2} \eta_{ab} \dot{\varphi}^a \dot{\varphi}^b - V \} \tag{3.51}$$

with the potential (3.5) rewritten in the new variables

$$V = \Lambda \exp[2q\varphi^0] + \exp(2q^{-1}\varphi^0) V_*(\vec{\varphi}_*), \tag{3.52}$$

where $V_*(\vec{\varphi}_*)$ is defined in (3.46), the vectors $\vec{u}^i_* \in \mathbf{R}^{n-1}$ satisfy the relations (3.44) with $N_0 = 1$, and $\vec{\varphi}_* = (\varphi_1, \ldots, \varphi_{n-1})$.

4 Exact solutions

Here we consider the metric (2.2) defined on the manifold (2.1) with the relations (2.4) and

$$M_0 = \mathbf{R}^{N_0}, \qquad g^0 = \sum_{a=1}^{N_0} dx^a \otimes dx^a, \tag{4.1}$$

assuming $N_0 > 2$. Thus the N_0-dimenional section of the metric (2.2) is conformally flat. One of the simplest Ansätze for (2.2) is the following:

$$\gamma = \alpha_0 u(|x|^2), \qquad \phi^i = \alpha_i u(|x|^2) + \beta_i, \tag{4.2}$$

where $\alpha_0, \alpha_i, \beta_i$ are constants, $i = 1, \ldots, n$, and $|x|^2 = \sum_{a=1}^{N_0} (x^a)^2$. We are interested in spherically symmetric solutions to the Einstein equations (2.5) with $\Lambda = 0$ governed by the function $u = u(z)$ and the parameters α_ν, β_i. The field equations

$$R_{MN}[g] = 0 \tag{4.3}$$

for the metric (2.2) satisfying (4.1) and (4.2), are equivalent to the following set of equations:

$$A \equiv -\alpha_0 (4zu'' + 2N_0 u') + 4\alpha_0 \hat{\alpha} z (u')^2 + 2\hat{\alpha} u' = 0, \tag{4.4}$$

$$B \equiv \hat{\alpha} u'' + [(N_0 - 2)\alpha_0^2 + 2\alpha_0 \sum_{j=1}^{n} N_j \alpha_j - \sum_{j=1}^{n} N_j \alpha_j^2](u')^2 = 0, \tag{4.5}$$

$$C_i \equiv \lambda_i - \alpha_i \,\mathrm{e}^{2(\alpha_i - \alpha_0)u + 2\beta_i} \times [4zu'' + 2N_0 u' - 4\hat{\alpha} z (u')^2] = 0, \tag{4.6}$$

$i = 1, \ldots, n$. Here $u' = du/dz$, $u'' = d^2u/dz^2$ and

$$\hat{\alpha} = (2 - N_0)\alpha_0 - \sum_{j=1}^{n} N_j \alpha_j. \tag{4.7}$$

The reduction of (4.3) to Eqs. (4.4)-(4.6) takes place due to the following representation for the Ricci tensor components (2.8) and (2.9) in our case (4.2):

$$R_{ab}[g] = A\delta_{ab} + 4Bx^a x^b, \tag{4.8}$$

$$R_{m_i n_i}[g] = C_i g^i_{m_i n_i}, \tag{4.9}$$

$a, b = 1, \ldots, N_0$; $i = 1, \ldots, n$.

Here we adopt the following Ansatz for the function $u(z)$ from (4.2):

$$u(z) = \ln(C + z), \tag{4.10}$$

where C is a constant. Under the substitution (4.10) Eq. (4.4) is satisfied identically if

$$\hat{\alpha} = -1, \qquad \alpha_0 = -1/N_0. \tag{4.11}$$

(We note that $u'' = -(u')^2$. For $C \neq 0$, (4.4) implies (4.11).) Then, (4.4) and (4.5) read:

$$\sum_{j=1}^{n} N_j \alpha_j = 2 - \frac{2}{N_0}, \tag{4.12}$$

$$\sum_{j=1}^{n} N_j \alpha_j^2 = \frac{(N_0 - 1)(N_0 - 2)}{N_0^2}. \tag{4.13}$$

Eqs. (4.6) are equivalent to the relations

$$2(\alpha_0 - \alpha_i) = -1, \qquad 2N_0 \alpha_i \, e^{2\beta_i} = \lambda_i, \tag{4.14}$$

$i = 1, \ldots, n$. From (4.11) and (4.14) we obtain

$$\alpha_i = \frac{1}{2} - \frac{1}{N_0}, \qquad e^{2\beta_i} = \frac{\lambda_i}{N_0 - 2} \neq 0. \tag{4.15}$$

A substitution of (4.15) into (4.12), (4.13) gives the following Diophantus equation for the dimensions N_ν:

$$\sum_{j=1}^{n} N_j = \frac{4(N_0 - 1)}{N_0 - 2}. \tag{4.16}$$

Eq. (4.16) has the solutions

$$\sum_{j=1}^{n} N_j = 8, 6, 5 \quad \text{for} \quad N_0 = 3, 4, 6, \tag{4.17}$$

respectively. ¿From (2.2), (4.1), (4.2), (4.10), (4.11) and (4.15) we obtain the metric

$$g = \left[C + |x|^2\right]^{1-2/N_0} \left[\sum_{a=1}^{N_0} \frac{dx^a \otimes dx^a}{C + |x|^2} + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i\right] \tag{4.18}$$

defined on the manifold

$$M = \mathbf{R}_C^{N_0} \times M_1 \times \ldots \times M_n, \tag{4.19}$$

where

$$\mathbf{R}_C^{N_0} = \{x \in \mathbf{R}^{N_0} : C + |x|^2 > 0\} \subset \mathbf{R}^{N_0} \tag{4.20}$$

is an open domain in $\mathbf{R}^{N_0}$, $C \in \mathbf{R}$. The metric (4.18) describes, for $N_0 = 3, 4, 6$, three families of spherically symmetric ($O(N_0)$-symmetric) solutions to the vacuum Einstein equations (4.3) with n internal Einstein spaces of nonzero curvature (M_i, g^i) (2.4). It follows from (4.16), (4.17) that

$$D = N_0 + \sum_{j=1}^{n} N_j = \frac{N_0^2}{N_0 - 2} + 2 = 11, 10, 11, \tag{4.21}$$

$$n \leq n_0 = 4, 3, 2 \tag{4.22}$$

for $N_0 = 3, 4, 6$, respectively.

4.1. Nonsingular solutions

For $C > 0$, $\mathbf{R}_C^{N_0} = \mathbf{R}^{N_0}$ and the metric (4.18) describes spherically symmetric nonsingular solutions to the Einstein equations defined on the manifold

$$\mathbf{R}^{N_0} \times M_1 \times \ldots \times M_n. \tag{4.23}$$

(It should be stressed that the N_0-dimensional part of the metric (4.18) has Euclidean signature.) A special case of this solution with $N_0 = 6$, $n = 1$, $N_1 = 5$ was recently considered in [42].

4.2. Exceptional solutions

Let us consider the solution (4.18) with $C = 0$. It can be written as follows:

$$g = d\rho \otimes d\rho + \rho^2 g_*, \qquad \rho = \alpha^{-1} |x|^{\alpha} \tag{4.24}$$

where $\alpha = 1 - 2/N_0$ and

$$g_* = \alpha^2 \left[g(S^{N_0 - 1}) + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i\right] \tag{4.25}$$

is the Einstein metric on the manifold

$$M_* = S^{N_0 - 1} \times M_1 \times \ldots \times M_n. \tag{4.26}$$

Here $g(S^{N_0-1})$ is the canonical metric on an $(N_0{-}1)$-dimensional sphere S^{N_0-1}. The metric g_* in (4.24) satisfies the relation

$$\mathrm{Ric}\,[g_*] = (D - 2) g_*, \tag{4.27}$$

where $\mathrm{Ric}\,[g_*]$ is the Ricci tensor corresponding to g_* and $D = \dim M$. The metric (4.24) is defined on the manifold $\mathbf{R}_+ \times M_*$ (see Remark 1) and is non-flat, as may be verified using the relations (6.2)-(6.4) from the Appendix. The N_0-dimensional section of the metric is also non-flat (due to "deficit" of the spherical angle). Since the solution (4.24) is an attractor for (4.18) as $|x| \to \infty$, we see that the metric (4.18) and its N_0-dimensional section have non-flat asymptotics.

4.3. Solutions with arbitrary signature

The solution (4.18) may be considered as a special case of the following solutions with arbitrary signature of "our" space:

$$g = \left[C + \eta_{ab} x^a x^b\right]^{1-2/N_0} \left\{ \frac{\eta_{ab} dx^a \otimes dx^b}{C + \eta_{ab} x^a x^b} + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i \right\}. \tag{4.28}$$

Here

$$\eta = (\eta_{ab}) = \mathrm{diag}(w_1, \ldots, w_{N_0}), \qquad w_a = \pm 1. \tag{4.29}$$

The metric (4.28) is defined on the manifold

$$M = \mathbf{R}_{C,\eta}^{N_0} \times M_1 \times \ldots \times M_n, \tag{4.30}$$

where

$$\mathbf{R}_{C,\eta}^{N_0} = \{x \in \mathbf{R}^{N_0} : C + \eta_{ab} x^a x^b > 0\} \subset \mathbf{R}^{N_0} \tag{4.31}$$

is supposed to be non-empty (i.e the case when $C < 0$ and all $w_a = -1$ in (4.29) is excluded). The metric (4.28) satisfies the vacuum Einstein equations (4.3). It may be obtained from (4.18) by a Wick-type rotation, i.e. we write $x^a = w_a^{1/2} \hat{x}^a$, $w_a > 0$, in (4.18) and then perform an analytical continuation in w_a.

Proposition 1. The Riemann tensor squared for the metric (4.28) has the form

$$
\begin{aligned}
I[g] &\equiv R_{MNPQ}[g]R^{MNPQ}[g] \\
&= (C+x^2)^{-2-2\alpha}(\bar{I}_1+\bar{I}_2),
\end{aligned} \tag{4.32}
$$

where

$$
\begin{aligned}
\bar{I}_1 = {}& (\alpha-1)^2(N_0-1)\{16C^2 \\
& + 2(N_0-2)[2C+(\alpha+1)x^2]^2\},
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
\bar{I}_2 = {}& -4\alpha^2 N(N_0-2)x^2(C+x^2) \\
& + (C+x^2)^2\sum_{i=1}^{n}\Big(\frac{N_0-2}{\lambda_i}\Big)^2 I[g^i] + 2\alpha^4 N(N-1)(x^2)^2 \\
& + 4\alpha^2 N(N_0-1)(\alpha x^2+C)^2;
\end{aligned} \tag{4.34}
$$

here $\alpha = 1-2/N_0$, $x^2=\eta_{ab}x^a x^b$, $N=\sum_{j=1}^n N_j$ and $I[g^i]$ is the Riemann tensor squared for the metric g^i.

Proof. Eqs. (4.32)-(4.34) may be obtained using the formula (6.10) from the Appendix. But a simpler way is to calculate first the Riemann tensor squared in the Euclidean case $\eta_{ab}=\delta_{ab}$,

$$
g = [C+r^2]^{\alpha}\Big\{\frac{dr\otimes dr + r^2 d\Omega^2_{N_0-1}}{C+r^2} + \sum_{i=1}^{n}\frac{\lambda_i}{N_0-2}\, g^i\Big\} \tag{4.35}
$$

where $r^2=\delta_{ab}x^a x^b$ and $d\Omega^2_{N_0-1}=g(S^{N_0-1})$ is the metric on S^{N_0-1}, using the "cosmological" relation (6.15) from the Appendix, and then perform the Wick rotation $r^2\to\eta_{ab}x^a x^b$.

Proposition 2. For the metric (4.28) with a non-Euclidean signature $(\eta_{ab})\neq(\delta_{ab})$ and $C\neq 0$

$$
R_{MNPQ}[g]R^{MNPQ}[g]\to +\infty \tag{4.36}
$$

as $C+\eta_{ab}x^a x^b\to +0$.

Proof. From (4.32)-(4.34) we obtain

$$
R_{MNPQ}[g]R^{MNPQ}[g]\sim A_1[C+\eta_{ab}x^a x^b]^{-2-2\alpha} \tag{4.37}
$$

as $C+\eta_{ab}x^a x^b\to +0$, where

$$
\begin{aligned}
A_1 = {}& (\alpha-1)^2(N_0-1)C^2[16+2(N_0-2)(1-\alpha)^2] \\
& + 2N\alpha^2 C^2[2+(N-1)\alpha^2+2(N_0-1)(1-\alpha)^2] > 0.
\end{aligned} \tag{4.38}
$$

Then (4.36) follows from (4.37), (4.38) and $\alpha>0$.

Thus the solution (4.28) with a non-Euclidean signature $\eta=(\eta_{ab})\neq(\delta_{ab})$ and $C\neq 0$ cannot be extended to the manifold (4.23).

For N_0=4, $\sum_{i=1}^n N_i=6$, $\eta=\pm\mathrm{diag}(-1,1,1,1)$, we get an $O(1,3)$-symmetric solution in 10-dimensional gravity with a pseudo-Euclidean conformally flat 4-dimensional section

$$
g = [C\pm x^2]^{1/2}\Big\{\frac{-dx^0\otimes dx^0 + d\vec{x}\otimes d\vec{x}}{\pm C+x^2} + \sum_{i=1}^{n}\frac{\lambda_i}{N_0-2}g^i\Big\}, \tag{4.39}
$$

where $x^2=-(x^0)^2+(\vec{x})^2$.

Remark 4. The "Euclidean" solution (4.35) with $C=1$ may be also written in the form

$$
g = (\cosh y)^{2\alpha}\Big\{dy\otimes dy + \tanh^2 y\, d\Omega^2_{N_0-1} + \sum_{i=1}^{n}\frac{\lambda_i}{N_0-2}g^i\Big\}, \tag{4.40}
$$

where $\sinh y = r$ and $\alpha=1-2/N_0$). The N_0-dimensional section of (4.40) contains a "sigar-type" metric multiplied by a conformal factor:

$$
g_s = (\cosh y)^{2\alpha}\{dy\otimes dy + \tanh^2 y\; d\Omega^2_{N_0-1}\}. \tag{4.41}
$$

De Sitter membrane. Let $n=1$ and

$$
g^1 = g(dS^{N_1}) = -dt\otimes dt + \frac{\cosh^2(Ht)}{H^2}d\Omega^2_{N_1-1} \tag{4.42}
$$

be the N_1-dimensional de Sitter metric, where N_1 is defined in (4.16) and

$$
H^2 = \frac{N_0-2}{N_1-1} = \frac{(N_0-2)^2}{3N_0-2}. \tag{4.43}
$$

The metric (4.42) satisfies the relation

$$
\mathrm{Ric}\,[g(dS^{N_1})] = (N_0-2)\; g(dS^{N_1}), \tag{4.44}
$$

and hence the metric

$$
g = [C+r^2]^{\alpha}\Big\{\frac{dr\otimes dr + r^2\Omega^2_{N_0-1}}{C+r^2} - dt\otimes dt + \frac{\cosh^2(Ht)}{H^2}d\Omega^2_{N_1-1}\Big\}, \tag{4.45}
$$

($\alpha=1-2/N_0$) satisfies the Einstein equations. The metric (4.45) describes a spherically symmetric nonsingular de Sitter membrane solution.

The curvature-splitting trick. The solution (4.28) with n internal spaces may be obtained from the one with $n=1$ by so-called "curvature-splitting" trick [41]. Let us consider a set of k Einstein manifolds $(\mathcal{M}_i, h^i)$ of nonzero curvature, i.e.

$$
\mathrm{Ric}\,(h^i) = \mu_i h^i, \tag{4.46}
$$

where $\mu_i\neq 0$ is a real constant, $i=1,...,k$. Let $\mu\neq 0$ be a real number. Then

$$
h = \sum_{i=1}^{k}\frac{\mu_i}{\mu}h^i \tag{4.47}
$$

is an Einstein metric, (correctly) defined on

$$\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_k \tag{4.48}$$

and satisfying

$$\mathrm{Ric}\ (h) = \mu h. \tag{4.49}$$

Indeed,

$$\begin{aligned} \mathrm{Ric}\ (h) &= \sum_{i=1}^{k} \mathrm{Ric}\ \left(\frac{\mu_i}{\mu} h^i\right) \\ &= \sum_{i=1}^{k} \mathrm{Ric}(h^i) = \sum_{i=1}^{k} \mu_i h^i = \mu h. \end{aligned} \tag{4.50}$$

(Here we have simplified the notations according to Remark 1.)

5 The case $N_0 = 2$

Consider now the exceptional case $N_0 = 2$. In this case the action (2.12) reads (we put here $\kappa_0^2 = 1$)

$$\begin{aligned} S &= S_\sigma[g^0, \gamma, \phi] \\ &= \frac{1}{2} \int_{M_0} d^2x \sqrt{|g^0|} \exp\left(\sum_{i=1}^{n} N_i \phi^i\right) \Big\{ R[g^0] \\ &\quad - \bar{G}_{ij}(\partial\phi^i)(\partial\phi^j) + 2(\partial\gamma) \sum_{j=1}^{n} N_j \partial\phi^j \\ &\quad + \sum_{i=1}^{n} \lambda_i N_i \,\mathrm{e}^{-2\phi^i + 2\gamma} - 2\Lambda \,\mathrm{e}^{2\gamma} \Big\}, \end{aligned} \tag{5.1}$$

where $\bar{G}_{ij}$ is the cosmological minisuperspace metric (3.49). ¿From (5.1) we see that the midisuperspace metric crucially depends upon the choice of γ. For $\gamma = 0$ we get from (5.1) the action with a conformally flat midisuperspace metric of pseudo-Euclidean signature

$$\begin{aligned} S &= \frac{1}{2} \int_{M_0} d^2x \sqrt{|g^0|} \exp\left(\sum_{i=1}^{n} N_i \phi^i\right) \Big\{ R[g^0] \\ &- \bar{G}_{ij}(\partial_\mu\phi^i)(\partial_\nu\phi^j) g^{0\mu\nu} + \sum_{i=1}^{n} \lambda_i N_i \,\mathrm{e}^{-2\phi^i} - 2\Lambda \Big\}. \end{aligned} \tag{5.2}$$

Another choice of the conformal frame parameter

$$\gamma = -\frac{1}{2} \sum_{i=1}^{n} N_i \phi^i \tag{5.3}$$

leads us to the action

$$\begin{aligned} S &= \frac{1}{2} \int_{M_0} d^2x \sqrt{|g^0|} \exp\left(\sum_{i=1}^{n} N_i \phi^i\right) \Big\{ R[g^0] \\ &- \sum_{i=1}^{n} N_i (\partial_\mu\phi^i)(\partial_\nu\phi^i) g^{0\mu\nu} \\ &+ \left(\sum_{i=1}^{n} \lambda_i N_i \,\mathrm{e}^{-2\phi^i} - 2\Lambda\right) \exp\left(-\sum_{i=1}^{n} N_i \phi^i\right) \Big\}, \end{aligned} \tag{5.4}$$

with a Euclidean conformally flat midisuperspace metric. Note that in Ref. [3] the action (5.2) was reduced to a "string-like" form (for $n = 1$ see, for example, [53]).

6 Appendix

6.1. Riemann tensor.

Here we consider the metric

$$g = \bar{g}^0 + \sum_{i=1}^{n} \mathrm{e}^{2\phi^i(x)} g^i, \tag{6.1}$$

defined on the manifold (2.1), where the metrics $\bar{g}^0$ and g^i are defined on M_0 and M_i respectively, $i = 1, \ldots, n$. The nonzero components of the Riemann tensor corresponding to (6.1) are

$$R_{\mu\nu\rho\sigma}[g] = R_{\mu\nu\rho\sigma}[\bar{g}^0], \tag{6.2}$$

$$\begin{aligned} R_{\mu m_i \nu n_i}[g] &= -R_{m_i \mu \nu n_i}[g] = -R_{\mu m_i n_i \nu}[g] \\ = R_{m_i \mu n_i \nu}[g] &= -\,\mathrm{e}^{2\phi^i} g^i_{m_i n_i} [\bigtriangledown_\mu[\bar{g}^0](\partial_\nu\phi^i) \\ &\quad + (\partial_\mu\phi^i)(\partial_\nu\phi^i)], \end{aligned} \tag{6.3}$$

$$\begin{aligned} R_{m_i n_j p_k q_l}[g] &= \mathrm{e}^{2\phi^i} \delta_{ij}\delta_{kl}\delta_{ik} R_{m_i n_i p_i q_i}[g^i] \\ &+ \mathrm{e}^{2\phi^i + 2\phi^j} \bar{g}^{0\ \mu\nu} (\partial_\mu\phi^i)(\partial_\nu\phi^j)[\delta_{il}\delta_{jk} g^i_{m_i q_i} g^j_{n_j p_j} \\ &- \delta_{ik}\delta_{jl} g^i_{m_i p_i} g^j_{n_j q_j}], \end{aligned} \tag{6.4}$$

where the indices μ, ν, ρ, σ correspond to M_0, m_i, n_i, p_i, q_i to M_i; $i, j, k, l = 1, \ldots, n$, $\bigtriangledown[g^0]$ is a covariant derivative with respect to g^0.

The relations (6.2)-(6.4) may be obtained from the following relations for the nonzero components of the Christophel-Schwarz symbols:

$$\Gamma^\mu_{\nu\rho}[g] = \Gamma^\mu_{\nu\rho}[\bar{g}^0], \tag{6.5}$$

$$\Gamma^{m_i}_{n_i \nu}[g] = \Gamma^{m_i}_{\nu n_i}[g] = \delta^{m_i}_{n_i} \partial_\nu\phi^i, \tag{6.6}$$

$$\Gamma^\mu_{m_i n_i}[g] = -\bar{g}^{0\ \mu\nu} (\partial_\nu\phi^i)\,\mathrm{e}^{2\phi^i} g^i_{m_i n_i}, \tag{6.7}$$

$$\Gamma^{m_i}_{n_i p_i}[g] = \Gamma^{m_i}_{n_i p_i}[g^i]. \tag{6.8}$$

6.2. Riemann tensor squared.

We denote the squared Riemann tensor by

$$I[g] \equiv R_{MNPQ}[g] R^{MNPQ}[g]. \tag{6.9}$$

As follows from Eqs. (6.2)-(6.4), for the metric (6.1) [39]

$$\begin{aligned} I[g] = I[\bar{g}^0] &+ \sum_{i=1}^{n} \{ \,\mathrm{e}^{-4\phi^i} I[g^i] - 4\,\mathrm{e}^{-2\phi^i} U[\bar{g}^0, \phi^i] R[g^i] \\ &- 2N_i U^2[\bar{g}^0, \phi^i] + 4N_i V[\bar{g^0}, \phi^i] \} \\ &+ \sum_{i,j=1}^{n} 2N_i N_j [\bar{g}^{(0),\mu\nu} (\partial_\mu\phi^i)\partial_\nu\phi^j]^2, \end{aligned} \tag{6.10}$$

where $R[g^i]$ is the scalar curvature of g^i and $N_i = \dim M_i$, $i = 1, \ldots, n$. In (6.10)

$$U[g,\phi] \equiv g^{MN}(\partial_M\phi)\partial_N\phi, \tag{6.11}$$

$$\begin{aligned} V[g,\phi] \equiv\ & g^{M_1N_1}g^{M_2N_2} \times \\ & \times [\bigtriangledown_{M_1}(\partial_{M_2}\phi) + (\partial_{M_1}\phi)\partial_{M_2}\phi] \times \\ & \times [\bigtriangledown_{N_1}(\partial_{N_2}\phi) + (\partial_{N_1}\phi)\partial_{N_2}\phi], \end{aligned} \tag{6.12}$$

where $\bigtriangledown = \bigtriangledown[g]$ is a covariant derivative with respect to g.

6.3. The cosmological case

Consider now the special case of (6.10) with $M_0 = (t_1, t_2)$, $t_1 < t_2$. Thus we consider the metric

$$g_c = -B(t)dt \otimes dt + \sum_{i=1}^{n} A_i(t)g^i, \tag{6.13}$$

defined on the manifold

$$M = (t_1, t_2) \times M_1 \times \ldots \times M_n. \tag{6.14}$$

and $B(t), A_i(t) \neq 0$, $i = 1, \ldots, n$.

¿From (6.11) we obtain the Riemann tensor squared for the metric (6.13) [37, 39]

$$\begin{aligned} I[g_c] = \sum_{i=1}^{n} \Big\{ & A_i^{-2}I[g^i] + A_i^{-3}B^{-1}\dot{A}_i^2R[g^i] \\ & - \frac{1}{8}N_iB^{-2}A_i^{-4}\dot{A}_i^4 + \frac{1}{4}N_iB^{-2}\big(2A_i^{-1}\ddot{A}_i \\ & - B^{-1}\dot{B}A_i^{-1}\dot{A}_i - A_i^{-2}\dot{A}_i^2\big)^2\Big\} \\ & + \frac{1}{8}B^{-2}\left[\sum_{i=1}^{n} N_i(A_i^{-1}\dot{A}_i)^2\right]^2. \end{aligned} \tag{6.15}$$

6.4. Conformal transformation

We present for convenience the well-known relations [54]

$$\begin{aligned} \mathrm{e}^{-2\gamma}R_{\mu\nu\rho\sigma}[\mathrm{e}^{2\gamma}g^0] = {} & R_{\mu\nu\rho\sigma}[g^0] \\ & + Y_{\nu\rho}g^0_{\mu\sigma} - Y_{\mu\rho}g^0_{\nu\sigma} - Y_{\nu\sigma}g^0_{\mu\rho} + Y_{\mu\sigma}g^0_{\nu\rho}, \end{aligned} \tag{6.16}$$

$$R_{\mu\nu}[\mathrm{e}^{2\gamma}g^0] = R_{\mu\nu}[g^0] + (2-N_0)Y_{\mu\nu} - g^0_{\mu\nu}g^{0\rho\tau}Y_{\rho\tau}, \tag{6.17}$$

$$\triangle\,[\mathrm{e}^{2\gamma}g^0] = \mathrm{e}^{-2\gamma}\{\Delta_0 + (N_0-2)g^{0\mu\nu}(\partial_\mu\gamma)\partial_\nu\} \tag{6.18}$$

where, as in Subsec. 2.1, the metric g^0 is defined on M_0, $\dim M_0 = N_0$, Δ_0 is the Laplace-Beltrami operator on M_0 and

$$Y_{\mu\nu} = \gamma_{;\mu\nu} - \gamma_\mu\gamma_\nu + \tfrac{1}{2}g^0_{\mu\nu}\gamma_\rho\gamma^\rho. \tag{6.19}$$

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